The Mathematics of Emergence

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Outline

1. Complexity Science: my view
2. What is Emergence?
3. Space-time Phases
4. Quantifying Emergence
5. Strong Emergence
6. Questions about Space-time Phases
7. Conclusion

1. Complexity Science: my view

- The study of systems with many interdependent components
- e.g. laser, condensed matter, cell, brain, ecosystem, climate system, transport networks, internet, health service, finance, economy
- Hope for unifying principles: Mathematics

2. What is Emergence?

- Wikipedia: “Emergence is the way complex systems and patterns arise out of a multiplicity of relatively simple interactions”
- “The whole is more than the sum of its parts” (Aristotle, c330BC)
- “the whole becomes not merely more, but very different from the sum of its parts” (Anderson, 1972)

A “cathedral” mound produced by a termite colony: a classic example of emergence in nature.

Philosophers

- JS Mill: emergentism v. reductionism

"To whatever degree we might imagine our knowledge of the properties of the several ingredients of a living body to be extended and perfected, it is certain that no mere summing up of the separate actions of those elements will ever amount to the action of the living body itself." (*A System of Logic, Bk.III, Ch.6, §1, 1843)*

- Weak v. Strong emergence,

  e.g. Chalmers, 2006: A high-level phenomenon is weakly / strongly emergent with respect to a low-level domain when it arises from the latter but truths concerning it are

unexpected given the principles governing /

not deducible even in principle from truths in

My view: space-time phases

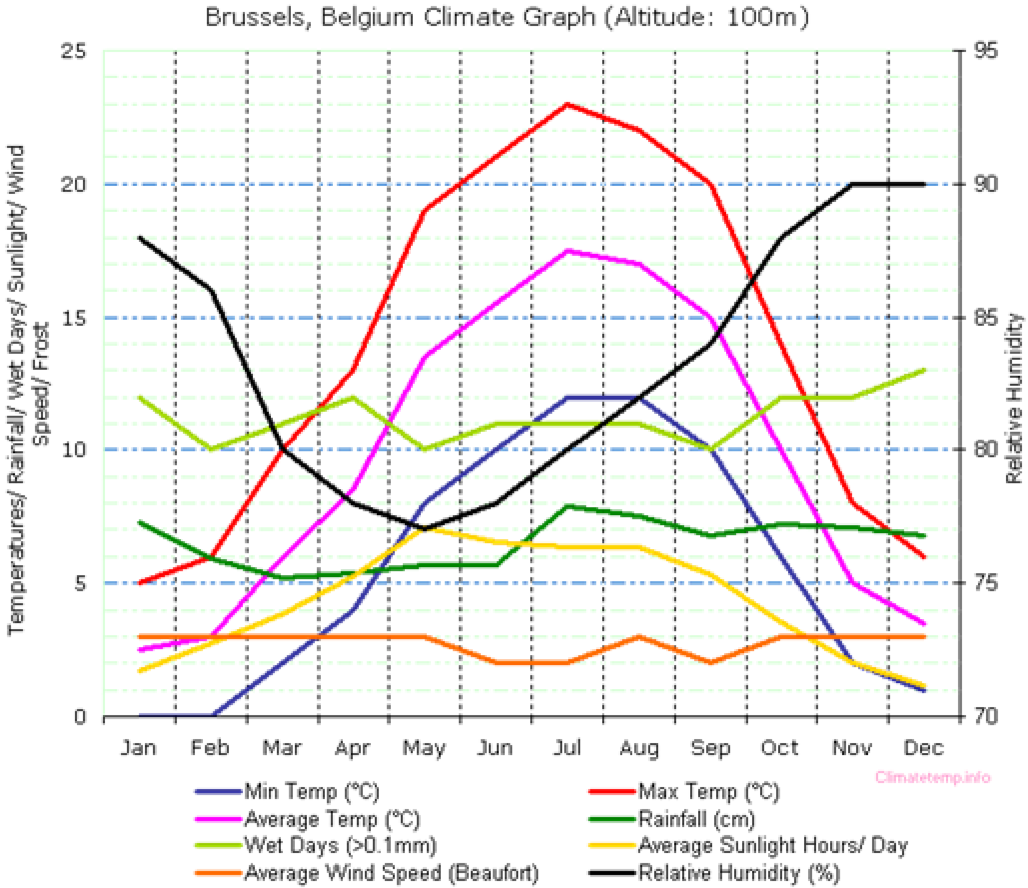
(a) What emerges from a spatially extended dynamical system is "space-time phases": probability distributions over realisations of state as function of space-time that arise from typical initial probabilities in the distant past.

(b) Amount of emergence is the "distance" of a space-time phase from the set of products for independent units.

(c) Strong emergence means non-unique space-time phase (but not due to decomposability).

3. Space-time Phases

- e.g. “Climate” is a probability distribution for temperature, precipitation etc over space-time, compatible with the laws of weather.

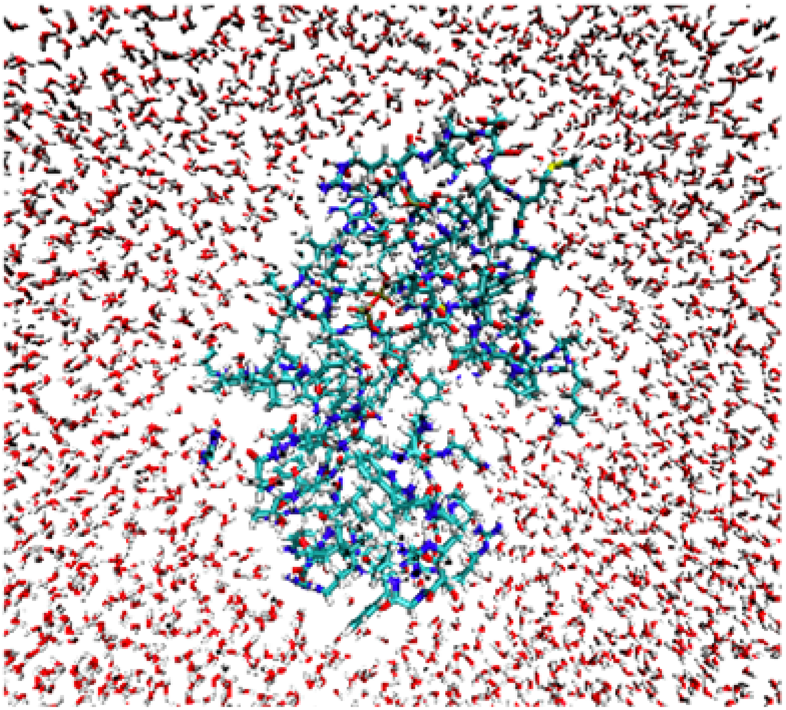


Equilibrium statistical mechanics

- The allowed probability distributions are the "Gibbs phases" for βH where H represents the sum of contributions h to the energy and β is coolness (1/temperature).
- i.e. probability density

$$\frac{1}{Z} e^{-\Sigma \beta h}$$

wrt reference measure, where Z is a normalisation constant, or better those whose conditionals for all finite subsystems and external states satisfy this (Dobrushin,Lanford,Ruelle).



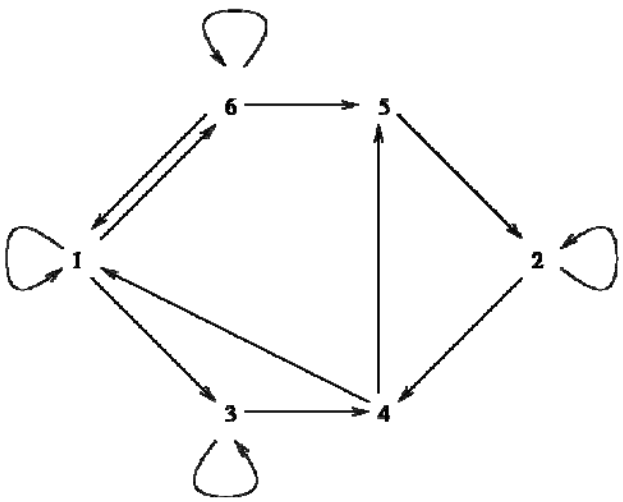
Dihydrofolate reductase in water (Dmitry Nerukh)

Stochastic dynamics

- For Markov chains the phases are the Gibbs phases (over time) for - log *p(i,j):* probability of sequence $i_0, i_1, \ldots, i_n$ =

$$\prod_t p(i_t, i_{t+1}) = e^{-\sum_t -\log p(i_t,\, i_{t+1})}$$

- Probabilistic cellular automata (PCA): update state σ_s^t at spatial site *s* and time *t* by independent probabilities conditional on current state σ^t
- Demonstration: Toom's NEC majority voter PCA with error rate $p = 0.15$, by Marina Diakonova.
- The phases of a PCA are the space-time Gibbs phases for – log $p(\sigma_s^{t+1} | \sigma^t)$ [Lebowitz, Maes, Speer].

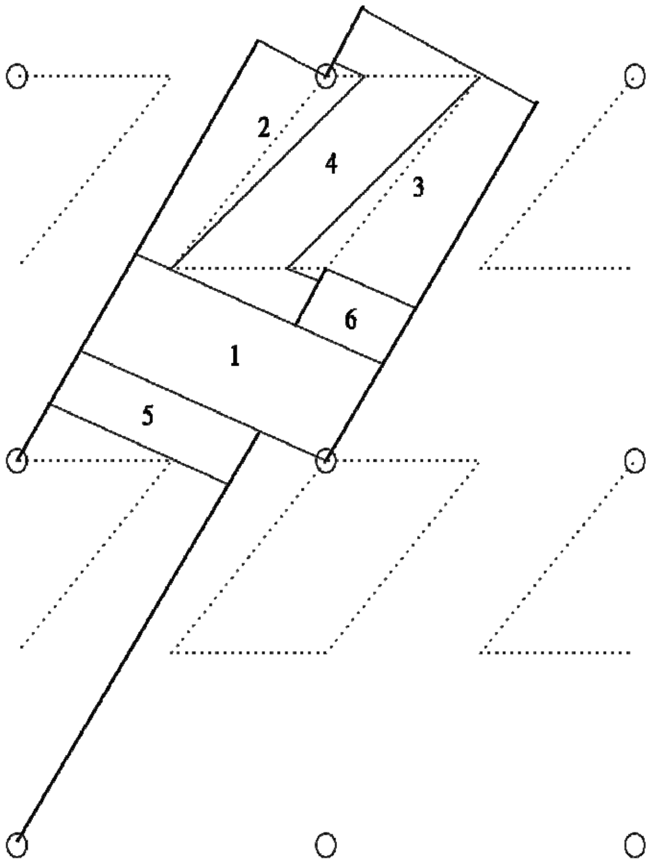


Deterministic dynamics

- Sensitive dependence on initial conditions makes individual trajectories unpredictable but often leads to a unique probability distribution on an attractor for random initial conditions in its basin.
- e.g. trajectories on a topologically mixing uniformly hyperbolic attractor for a map f can be coded by symbol strings σ, and random initial conditions in distant past in the basin give trajectories distributed according to unique Gibbs phase for

$$\beta H = \sum_t \log |\det Df_{E^-}(x^t(\sigma))|$$

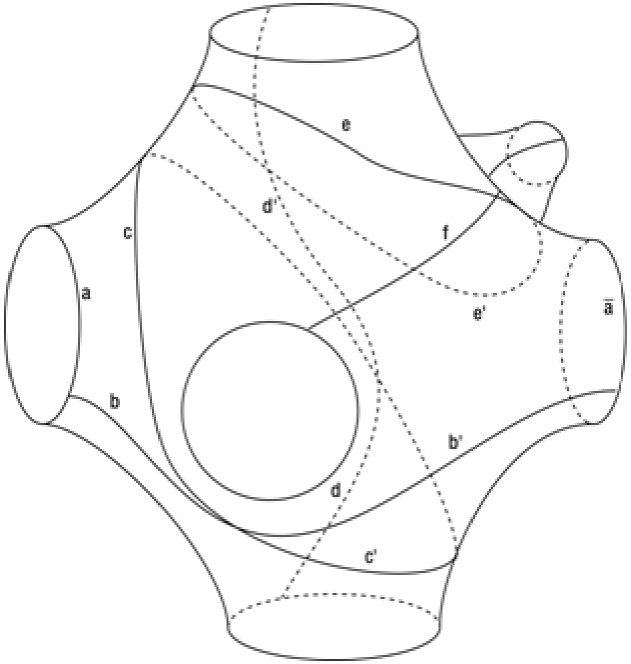
on time [Sinai, 1967/72]
- Analogous results for continuous-time.



Markov partition for Cerbelli-Giona map

A physical uniformly hyperbolic system

Show video



Minimal geodesics on configuration space from which to make a 40 element Markov partition

Spatially extended deterministic dynamics

- Trajectories of uniformly hyperbolic spatially extended discrete-time system *f* (coupled map lattice) can be coded by space-time symbol tables $\sigma = (\sigma_s^t)$ (Pesin & Sinai).
- Random initial conditions in distant past lead to distribution of trajectories given by Gibbs phases of

$$\beta H = \sum_{s,t} \mathrm{tr}(\log Df_{E^-}(x_s^{\,t}(\sigma)))_{ss}$$

(M,1995; Bricmont & Kupiainen, 1996).

4. Quantifying emergence

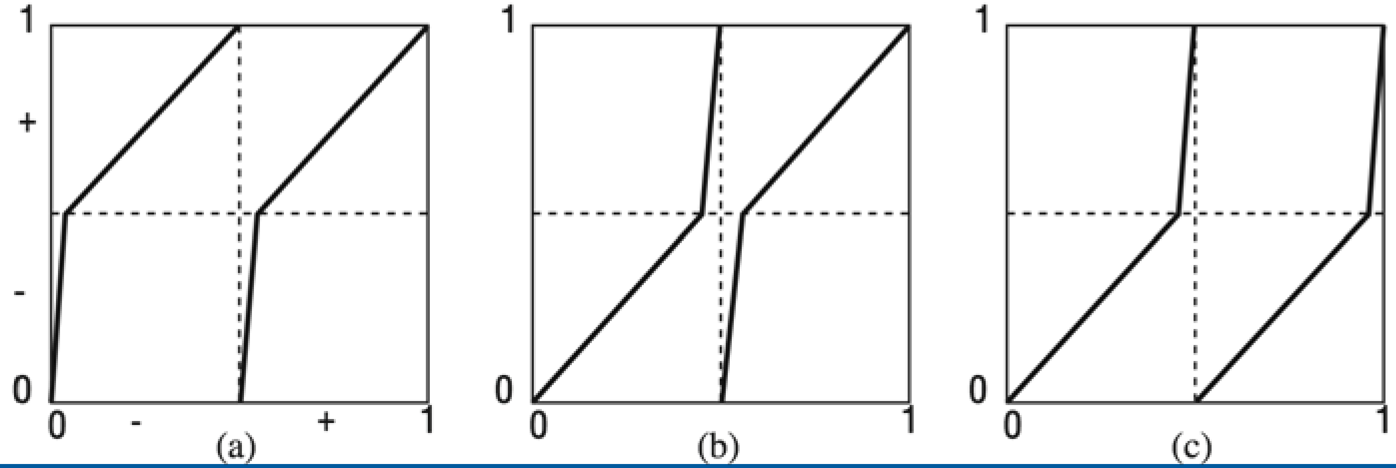
- Say the amount of emergence is the "distance" of a space-time phase from the set of product probabilities over independent units.
- For distance between multivariate probabilities I use a metric based on ideas of Dobrushin.
- So emergence measures how far the behaviour is from mean-field approximations.
- It does not capture what some people want to include in "emergence", e.g. law of averages, selection of Maxwellian velocity distribution,
- but does capture a likely consequence of having interdependent components.
- More interesting is to determine correlation structure of the phase, but it turns out to be an ingredient in the calculation of the Dobrushin distance.

5. Strong emergence

- More than one possible phase ("phase transition")
- Example: 2D Ising model (Peierls)
- Example: Toom's majority voter PCA with error rate 0.05
- Say amount of strong emergence is the diameter of the set of phases.
- Non-unique phase can arise for topological reasons, e.g. more than one attractor, or 2-piece attractor; more generally, because system is "decomposable". Don't count it as strong emergence.
- A system with a space-time symbolic description is "indecomposable" if any allowable configurations on two sufficiently separated space-time patches can be joined into an allowable configuration ("specification property").
- Non-trivial strong emergence requires infinite system, but is reflected in long-range correlations for finite versions.

Proved examples of strong emergence

- Ferromagnetic phases of 2D Ising model
- Ferromagnetic phases of Toom's NEC voter PCA
- Period-2 phases of Toom's NEC voter (error rate 0.95)
- Examples with (at least) 2^n extremal phases [demo], and also non-monotonic examples, e.g. 3 phases [demo]
- Endemic infection v disease-free phases of contact processes (Stavskaya…) [demo]
- Coupled map lattices based on these (Yamaguchi, Gielis&MacKay, Bardet&Keller)



6. Questions about space-time phases

Robustness

(i) How does a phase respond to a shock?

- Exponential decaying response to shocks in case of PCA with spectral gap, but more generally?

(ii) How does the phase or set of phases (closed, convex)vary with parameters?

- For PCA with a spectral gap, under small changes the unique phase stays unique and varies smoothly (cf. Ruelle for SRB measure of a uniformly hyperbolic dynamical system)
- For systems whose phases are Gibbsian, the set of phases varies upper hemi-continuously,
- but not always lower hemi-continuously, e.g. 2D Ising as magnetic field crosses 0.

Bifurcations

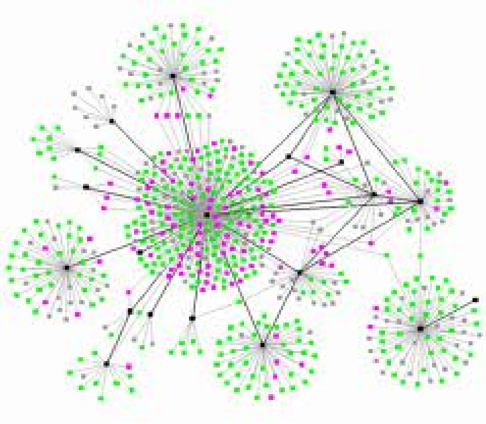
- In equilibrium statistical mechanics, co-existence of N (extremal) phases is of codimension N-1 (Gibbs phase rule),
- but for space-time phases, non-unique phase can be robust, e.g. Toom PCA.
- Does the set of phases generically vary smoothly? Perhaps there is a spectral projection that contains all the dynamics of domains?
- Proved examples of bifurcation: kinetic 2D Ising, Keller’s globally coupled maps
- Universality classes? Renormalisation (=aggregation + rescaling)?

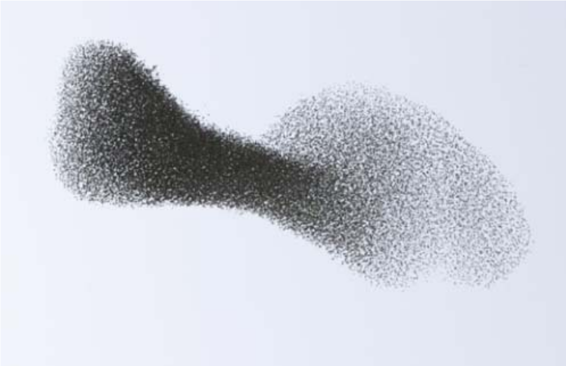
Control and Design

- What changes to a phase can be achieved by local control? A zealot can have huge effect in opinion-copying models [Mobilia, 2003]
- Boundary control can have a large effect when phase is non-unique [demo].
- What changes to the set of phases can be obtained with infinitesimal (but high gain) control? (cf. "control of chaos")
- How to design a complex system so that its phases optimise some objective function or partial order?

More questions

- More realistic systems, e.g. general network instead of a lattice, interaction of mobile units via proximity in space (swarms, flocks)?
- Special classes, e.g. multi-agent games, number-conserving systems, many-body quantum systems, quantum gravity?
- Systems that never settle down (evolution?)?
- Aggregation procedures
- Reduction to macroscopic models
- Fitting to data





7. Conclusion so far

- Complexity Science offers a lot of serious and worthwhile challenges.
- Complexity Science can benefit from serious mathematical input.
- But there is a huge gap between what mathematics can currently say and what users would like to know.
- Let’s work on it!

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Outline of the rest of the course

- Probabilistic cellular automata: regimes of unique phase, regimes of non-unique phase, Dobrushin metric
- Coupled map lattices: examples with non-unique phase, theory of space-time Gibbs phases