Nonlinear Dynamics and Complexity

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Analytic Problems

Maps

1. Consider the *logistic map* ($f_r^L(x)$) defined by

$$x_{n+1} = f_r^L(x_n) \equiv r x_n (1 - x_n) \text{ for } 0 < x_n < 1 \ ; \ 0 < r < 4 :$$

Verify that the period 2-orbit which is a fixed point of the second iterate $f_r^L(f_r^L(x^*)) = x^*$ satisfies the equation :

$$x^* \Big(x^* - (1 - \frac{1}{r}) \Big) \Big(x^{*2} - (1 + \frac{1}{r}) x^* + \frac{(1 + \frac{1}{r})}{r} \Big) = 0$$

Show that its solution for $r > 3$ is

$$x^*_{\pm}(r) = \frac{r+1}{2r} \pm \frac{1}{2r} \sqrt{(r+1)(r-3)}$$

Draw in a bifurcation diagram the stable and unstable fixpoints $x^*(r)$ vs. r for the parameter range $0 < r < 3 + \epsilon$.

2. We want to develop a cellular automaton (CA) approximation to the logistic map $x_{n+1} = r x_n (1 - x_n)$ by dividing the unit interval into eight bins, each of size $\frac{1}{8}$. By considering the eight values of x_n at the centers of the bins to be the eight possible states (call them s_i) of the CA – e.g., $s_1 = \frac{1}{16}$, $s_2 = \frac{3}{16}$, etc. – and by rounding off the exact result of the mapping to the nearest allowed CA state value, construct the "rule table" – i.e., the dynamical rule that takes the CA states at one time into the states at the next time for (a) $r = 2$; (b) $r = 3.2$; (c) $r = 3.6$; and (d) $r = 4$. How well does the (discretized state space) CA reflect the behavior of the (continuous state space) logistic map at the same values of r ?

What happens if we reduce the CA state space to four states by dividing the unit interval into four bins of size $\frac{1}{4}$ and study the same values of r ?

3. Consider the *tent map*, defined by $x_{n+1} = a[1 - 2(x_n - \frac{1}{2})]$. Prove that there is a mapping between the *logistic map* with $r = 4$ and the *tent map* for $a = 1$.

4. Consider the (dissipative) *Hénon map* for $-1 < \beta < +1$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = f^H_{\alpha,\beta} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \equiv \begin{pmatrix} 1 + \ y_n - \alpha x_n^2 \\ +\beta x_n \end{pmatrix}$$

For the parameters $.8 < \alpha < 1.05$ this map shows the same period doubling phenomenon as the logistic map this is an example of the universality of period doubling. Show that the Hénon map has real fixed points in the regime $\alpha > -\frac{1}{4}(\beta - 1)^2$ which satisfy

$$x^*_\pm = \frac{\beta - 1}{2\alpha} \pm \frac{1}{2\alpha}\sqrt{(\beta - 1)^2 + 4\alpha} \ \text{ and } \ \mathrm{y}^*_\pm = \beta \mathrm{x}^*_\pm$$

Prove that x^*_- is *unstable* while x^*_+ is *stable* if and only if $\alpha < \frac{3}{4}(1 - \beta)^2$.

For $\beta = .3$ and $\alpha \approx 1$ find the stable 2-orbit which corresponds to the fixed points of the second iterate $f^H_{\alpha,\beta}\left(f^H_{\alpha,\beta}\binom{x^*}{y^*}\right) = \binom{x^*}{y^*}$.

Draw the bifurcation diagram for the parameters $\beta = .3$ and $\alpha < 1$. Feel free to use the Mathematica DVD in an "experimental mathematics" approach to guide your analytic work.

5. Consider the *standard map* as discussed in class and defined by the equations

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = f^S_k \begin{pmatrix} p_n \\ q_n \end{pmatrix} \equiv \begin{pmatrix} p_n & - \frac{k}{2\pi} sin(2\pi q_n) \\ p_{n+1} & q_n \end{pmatrix}$$

, with p_n and Q_n both defined modulo 1 so the $p - q$ "plane" is actually a torus.Study the fixed points of this map and show that $p^* = 0$, $q^* = 0$ is a stable fixed point for $k < 4$ while $q^* = 1/2$ is unstable for any $k > 0$.

Show that for $p^* = \frac{1}{2}$ there is a periodic two orbit in which q^* oscillates from 0 to 1/2 to $1 = 0$ (mod 1). Study the stability of this limit cycle.

Low Dimensional Flows: ODEs

6. Relate the expression for the period of the simple pendulum

$$T = 4 \int_0^{\theta_{max}} \frac{d\theta}{\sqrt{2\omega^2 (cos\theta - cos\theta_{max})}}$$

to elliptic integrals. Show that $\frac{dT}{d\theta_{max}} > 0$ and that $\theta_{max} \rightarrow \pi$ as $T \rightarrow \infty$. A good source for information on elliptic functions is M. Abramowitz and I. Stegun, Handbook of Mathematical Functions, pp. 587 et. seq.

7. For the simple pendulum study the motion along the separatrix and show that the solution that has $\theta(t = 0) = 0$ is given by

$$\theta_{sx}(t) = (4tan^{-1}e^{+wt}) - \pi \ .$$

Comment qualitatively on the "stability" of this separatrix solution.

8. For the damped SHO verify and work out the analytical solutions for the cases (write the solutions in terms of the nondimensional quantity $r = \frac{\gamma}{2\omega_0}$)

   a) overdamping $\gamma > 2\omega_0$

   b) underdamping $\gamma < 2\omega_0$

   c) for the degenerate case $\gamma = 2\omega_0$ show, by taking the limit $\gamma \rightarrow 2\omega_0$ from b), that the solution is a product of an exponential and a linear function of time:

$$x(t) = e^{-\omega_0 t}[(\dot{x}(0) + \omega_0 x(0))t + x(0)]$$

9. Draw the phase plane for the quartic anharmonic potential $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$ which gives rise to the equation of motion $\ddot{x} - x + x^3 = 0$.

   a) Show that the origin is unstable and that the minima of the potential correspond to stable fixed points. Are there any separatrices?

   b) Add linear damping $\ddot{x} + \gamma\dot{x} - x + x^3 = 0$. Discuss qualitatively, using the phase plane, the resulting attractors and basins of attraction.

10. Consider a different variant of the quartic anharmonic potential in which both the quartic terms and the damping can be treated as small parameters. The equation of motion is thus $\ddot{u} + 2\varepsilon\mu\dot{u} + \omega_0^2 u + \varepsilon u^3 = 0$, with the assumption that $|\varepsilon| << 1$. Using multiple scale analysis, derive to first-order in ε a uniform expansion for u. [Hint: Be careful of the phase variation in the zeroth-order solution.]

High-Dimensional Flows: PDEs

11 In class we showed that the traveling wave solutions to the Korteweg-de Vries equation could be found by reducing it to an ODE in the co-moving frame $\zeta \equiv x - vt$:

$$-vu_\zeta + uu_\zeta + u_{\zeta\zeta\zeta} = 0$$

We studied the case in which all the constants of integration are zero so that the solution corresponds to a single soliton $u_s(x,t) = 3v\text{sech}^2\big(\sqrt{\frac{v}{2}}(x-vt)\big)$. Consider the case when the first constant of integration is zero but not the second; express this general solution in terms of elliptic functions.

12. Consider the **Toda lattice**, which as we discussed in class is an example for a completely integrable discrete system with N degrees of freedom:

$$M\ddot{y}_n = a\big(e^{-b(y_n - y_{n-1})} - e^{-b(y_{n+1}-y_n)}\big)$$

Derive the continuum limit of the Toda lattice assuming that the lattice spacing $a \to 0$ as well as the parameter $b \to 0$ while the sound speed c_0 – how should it be defined ? – remains constant. Keep the leading terms beyond the free linear wave equation and discuss the nature of the continuum limit.

13. Consider the linear dispersive wave equation

$$\Phi_t + c\Phi_x + \Phi_{xxx} = 0\ .$$

a) Consider a localized lump $\Phi(x,0) = e^{-\frac{x^2}{2}}e^{ik_0x}$ as the initial condition. Find its time evolution $\Phi(x,t)$ and examine the phase and group velocities $v_p \equiv \frac{\omega}{k}$, $v_g \equiv \frac{d\omega}{dk}$.

b) Consider a solution with Fourier amplitude $A(k) = e^{-\frac{1}{2}(k-k_0)^2}$. Show how the corresponding solution in the x-space to the above dispersive wave equation behaves.

14. For the Nonlinear Schrödinger Equation (NLSE) show by direct differentiation that

$$\Psi_S(x,t) = \Psi_o\frac{\exp[\frac{i}{2}v_1(x-v_2t)]}{\cosh[\sqrt{\frac{\kappa}{2}}\Psi_o(x-v_1t)]} \quad \text{where} \quad \Psi_o = \sqrt{\frac{v_1(v_1-2v_2)}{2\kappa}}$$

is a one-soliton solution.

15. Show that

$$\Phi_{K\overline{K}}(x,t) = 4\tan^{-1}\left(\frac{\sinh(\gamma vt)}{v\cosh(\gamma x)}\right)$$

is a "kink/anti-kink" solution to the sine-Gordon equation and plot its form for $t \in (-T, 0, +T)$ for a large $T \gg 0$. Show LAO that the sine-Gordon equation has a "breather" solution given by

$$\Phi_B(x,t) = 4\tan^{-1}\left(\frac{\epsilon \sin\left(\frac{t}{\sqrt{1+\epsilon^2}}\right)}{\cosh\left(\frac{\epsilon x}{\sqrt{1+\epsilon^2}}\right)}\right).$$

(Hint: observe the fact that the argument is of the form $\frac{f(t)}{g(x)}$ and use this to simplify your differentiation.) Plot the form of the breather for different phases of oscillations. Can you find a relation between these two solutions? [Hint: think of complex variables.]

16. Show that the function

$$u(x,t) = 72\left(\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{\left(3\cosh(x - 28t) + \cosh(3x - 36t)\right)^2}\right).$$

is an exact solution to the KdV equation. Show that this solution can be interpreted as a *two*-soliton solution by analyzing its asymptotic behavior. In particular, introduce variables $\xi = x - 16t$ and $\eta = x - 4t$, and show that as $t \to \pm\infty$ the solution consists entirely of two solitons, centered near ξ and $\eta = 0$. Describe the evolution of the solution from $t \to -\infty$ to $t \to +\infty$.