Nonlinear Dynamics and Complexity

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Computer-aided studies

The following problems are intended to be solved using the *Mathematica* Nonlinear DVD Notebook that was distributed to each of you. These specific problems are just meant to get you started. You are encouraged to explore different dynamical systems of your own creation; in essence, this is a series of exercises in what I called "experimental mathematics" –that is, the use of computers to gain qualitatively new insights, not simply to crunch numbers–in my first lecture.

Maps

1. Using *logistic map* notebook, calculate the *Lyapunov exponent* of the logistic map in the parameter region $2.9 < r < 3.5$. Can you explain the values observed in this range in terms of the analytic results derived in class for the *Lyapunov exponents* of periodic orbits ? In the region of the period-two orbit, compare the analytic results with the numerical ones.

2. Using the *"sine" logistic map* notebook, find the parameter values for the first four period-doubling bifurcations and use these to evaluate Feigenbaum's δ parameter. What value do you get? How does this compare with the "exact" result quoted in class? How does this compare to the result obtained in the "normal" logistic map, using the same approximation?

3. The *Hénon map* in the region $\beta = 0.3$ and $0.3 < \alpha < 1.05$ also undergoes a period-doubling transition to chaos as seen in the logistic map; this provides a simple example of the *universality* of period doubling in systems other than one-dimensional maps. Using the *Hénon map* notebook, study this region of

parameters and determine the values at which the various 2^n cycles first enter as functions of α at fixed $\beta = 0.3$. As in problem 2 above, try to get at least up to $n = 4$. From the data you gather, determine the Feigenbaum number δ defined in class. How well does your result agree with the exact answer given in class ? How does it compare with the results you obtained in problems 1 and 2 above?

4. Using the *standard map* notebook, consider the map for the value $k = \pi$. Can you find a stable period orbit ? If so, what is its period ? What values of p and q are involved ? Can you explain your results analytically ? For the same value of k, study the line $q = 0$ as a function of p for p near 0. Can you find other periodic orbits ?

5. Use the *standard map* notebook to study the single chaotic orbit that covers about half the phase space for $k = 1.1$. Start with $p = 0.01, q = 0.51$ and compare the results of 1000, 5000, and 10000 iterations. As discussed in class, this single orbit is a "fat fractal", occupying (as time approaches infinity) a finite fraction (roughly 0.56) of the total phase space area of 1.

Flows

6. Use the *pendulum* notebook to study the damped, driven plane pendulum. Vary the values of the damping and driving to see what different kinds of attractors exist in the system. (Hint: be careful to use an appropriate "absolute point size" so that you can see the attractors.) Be sure to study the section on periodic limit cycle attractors and print out the resulting graphics. [Hint: be very careful of the value of "PlotPoints," as large values can require considerable computer time.]

7. Use the *Lorentz equations* notebook to study the following questions. You will need to be careful about the limits of the plotting regions for the different ranges of parameters. Remembering that the fixed point value of z (whether the fixed point is stable or unstable) is $r-1$ is helpful. [Hint: this problem will take some time to solve, although the time will mostly be the computer's. Play with this just enough to get the idea.]

a. Determine the nature of the attracting set for $r = 350$. Study how it evolves as r is decreased in increments of 30 to a final value of 260. Use the x-z projection initially, but then study other projections.

[Hint: You will need to scale the x-z projection correctly so that the attracting orbit fits on the screen; for $r = 350$, try -60 to 60 for the x scale and 250 to 450 for the z scale. Be sure to eliminate transients by taking **TimeMin** close to **TimeMax** with their difference being large enough to contain at least one full period of the orbit. [That's a pretty big hint that the initial orbit is periodic!]

b. Study the period doubling window of the "xy" orbit by starting with $r = 240$ and working down to the "accumulation point" at $r = 214.364....$ From your best estimates of successive bifurcations, determine Feigenbaum's δ parameter.

c. Study the period doubling window of the "x^2y^2" for the range of r values between 145 and 166. Can you get another estimate of Feigenbaum's δ from this ?

d. If you do the linear stability analysis of the origin at the parameter values $\sigma = 10$, $\beta = 8/3$, and $r = 13.926$, you will find that the origin is *non*stable, with two stable directions given by

$$\vec{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \vec{s} = \begin{pmatrix} -0.77593 \\ 0.63081 \\ 0 \end{pmatrix}$$

and one unstable direction given by

$$\vec{u} = \begin{pmatrix} 0.50416 \\ 0.86361 \\ 0 \end{pmatrix}.$$

How are these directions determined ? Are these directions orthogonal ? Must they be ? What are the directions for the symmetry-related solution?